# SINGULAR PERTURBATIONS

by

John W. Searl, B.Sc.

Thesis submitted for the degree of

Doctor of Philosophy

of the

University of Edinburgh

October, 1969

## Summary

The nonlinear boundary value problem

$$\varepsilon x'' + F(t,x,x',\varepsilon) = 0, \quad 0 \leq t \leq 1,$$
$$a_1 x(0,\varepsilon) + \varepsilon a_2 x'(0,\varepsilon) = \alpha,$$
$$b_1 x(1,\varepsilon) + b_2 x'(1,\varepsilon) = \beta,$$

is examined under the hypothesis that an approximate solution $u(t,\varepsilon)$ exists in the sense that

$$\varepsilon u'' + F(t,u,u',\varepsilon) = H(t,\varepsilon) + H^*(t,\varepsilon), \quad 0 \leq t \leq 1$$

where $H(t,\varepsilon) = O(\eta)$, $H^*(t,\varepsilon) = O(\varepsilon^{-1}\eta\, e^{-m\phi/\varepsilon})$, $\eta = \eta(\varepsilon) = O(\varepsilon)$, $\phi = \phi(t,\varepsilon) = \int_0^t \phi'(s,\varepsilon)ds$ with $\phi'(t,\varepsilon)$ such that $F_{x'}(t,u,u',\varepsilon) = \phi'(t,\varepsilon) + O(\varepsilon)$,

and $m$ is a parameter ($\geq 1$) independent of both $t$ and $\varepsilon$. Under certain assumptions it is proved that for all sufficiently small $\varepsilon > 0$, the above boundary value problem possesses a solution $x(t,\varepsilon)$ such that

$$x - u = O(\eta) \quad \text{and} \quad x' - u' = O(\eta\,\Phi)$$

where
$$\Phi = \Phi(t,\varepsilon) = 1 + \frac{1}{\varepsilon} e^{-\phi(t,\varepsilon)/\varepsilon} \qquad \text{if } m > 1,$$
$$= 1 + \frac{1}{\varepsilon}\left(1 + \frac{\phi(t,\varepsilon)}{\varepsilon}\right)e^{-\phi(t,\varepsilon)/\varepsilon} \qquad \text{if } m = 1.$$

An example is given showing the construction of an approximate solution by the two-variable expansion method and its validification using the above result.

# Preface

The research described in this thesis was conducted under the supervision of Professor A. Erdélyi, Department of Mathematics.

## Contents

## Chapter 1

## An Introduction

We shall be concerned with the boundary value problem $P_\varepsilon$ consisting of the non-linear ordinary differential equation

$$(1.0) \qquad \varepsilon x'' + F(t,x,x',\varepsilon) = 0, \qquad 0 \leqslant t \leqslant 1$$

and the boundary conditions

$$(1.1) \qquad a_1\, x(0,\varepsilon) + \varepsilon a_2\, x'(0,\varepsilon) = \alpha(\varepsilon)\ ,$$

$$(1.2) \qquad b_1\, x(1,\varepsilon) + b_2 x'(1,\varepsilon) = \beta(\varepsilon)\ ,$$

where $\varepsilon$ is a small positive parameter. Two aspects of $P_\varepsilon$ make it worthy of study. The first concerns the existence of solutions. Coddington and Levinson [2] have furnished an example of a boundary value problem of the type $P_\varepsilon$ (with $F = x + x'^3$, $a_1 = b_1 = 1$, $a_2 = b_2 = 0$ and $\alpha$ and $\beta$ independent of $\varepsilon$) for which a solution exists only if $\varepsilon$ is greater than a certain bound depending on $|\alpha - \beta|$. Thus the continued existence of solutions as $\varepsilon \to 0$ is an important aspect of any investigation of $P_\varepsilon$. The second aspect concerns the behaviour of solutions as $\varepsilon \to 0$. If the limit

$$\lim_{\varepsilon \to 0+} x(t,\varepsilon)$$

exists we would expect it, where it is differentiable, to satisfy the reduced (or degenerate) differential equation

$$(1.3) \qquad F(t,u,u',0) = 0 \ .$$

But, because of the reduction in the order of the differential equation, we would expect a solution $u(t)$ of (1.3) to satisfy both (reduced) boundary conditions only exceptionally. Thus $x$ will approach its limiting value non-uniformly. To illustrate the situation consider the problem consisting of the non-linear differential equation

$$(1.5) \qquad \varepsilon x'' + xx' = 0, \quad 0 \leqslant t \leqslant 1$$

and the boundary conditions

$$(1.6) \qquad x(0,\varepsilon) = x_0\ , \quad x(1,\varepsilon) = x_1 \ .$$

Since/

Since the problem is non-linear, the dependence of the solution on its boundary conditions is non-trivial, so that apparently slight changes in $x_0$ and $x_1$ may result in important changes in the qualitative nature of $x$.

Firstly we take $x_0 = 0$ and $x_1 = 1$, and obtain a solution in the form

(1.7) $$x(t,\varepsilon) = A \tanh \frac{At}{2\varepsilon} \quad \text{with} \quad A \tanh \frac{A}{2\varepsilon} = 1 \ .$$

Setting $\varepsilon = 0$ in (1.5) we obtain the degenerate equation

$$u\,u' = 0 \ ,$$

with the solution $u = c$ (a constant). Letting $\varepsilon \to 0+$ in (1.7) we deduce

$$\lim_{\varepsilon \to 0+} x(t,\varepsilon) = 1, \quad 0 < t \leq 1 \ .$$

Thus choosing $c = 1$ we have $x(t,\varepsilon) \to u(t)$ as $\varepsilon \to 0$, $0 < t \leq 1$. We notice that the degenerate solution only satisfies the boundary condition at $t = 1$. The situation is shown in figure 1.

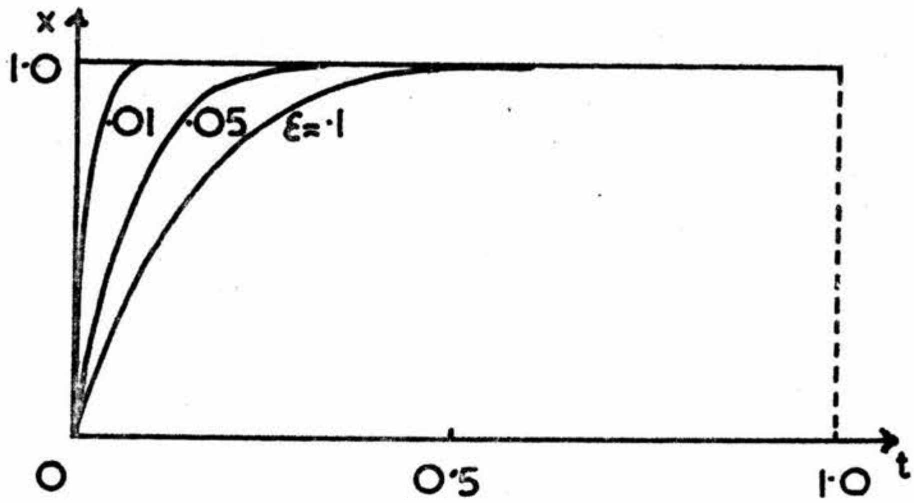


figure 1

$u(t)$ may be regarded as a uniform approximation to $x(t,\varepsilon)$, in the sense that $|x-u| = O(\varepsilon)$, in a domain $0 < t_0 \leq t \leq 1$, with $t_0$ independent of $\varepsilon$. The domain $0 \leq t \leq t_0$ where it ceases to be a uniform approximation is called the "boundary layer", after Prandtl's boundary layer theory for the singular perturbation problems arising

in/

in viscous flow. Such a boundary layer may occur at either end of the interval.

Next, we take $x_o = -1$ and $x_1 = +1$, and obtain

$$x(t,\varepsilon) = A \tanh\left(\frac{A}{2\varepsilon}(t-\tfrac{1}{2})\right) \quad \text{with} \quad A \tanh\frac{A}{4\varepsilon} = 1 \quad .$$

Here the boundary layer has broken away from the end of the interval to give a "shock layer", as shown in figure 2.

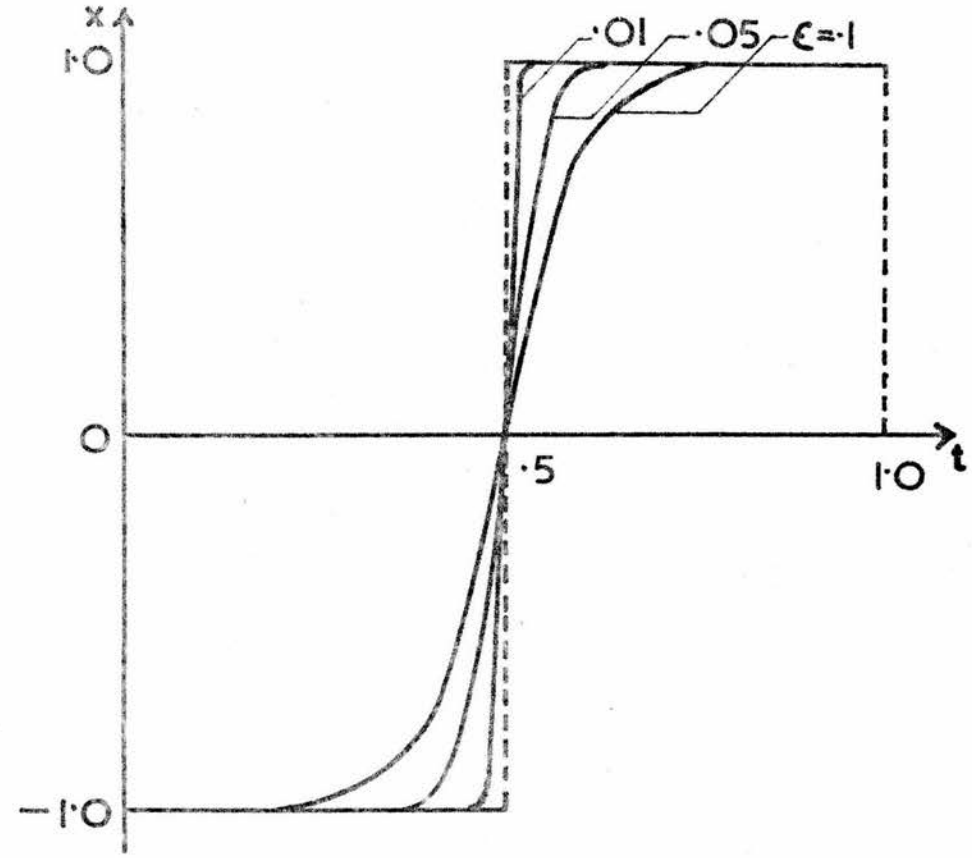


figure 2

Lastly, we take $x_o = +1$ and $x_1 = -1$, and obtain

$$x(t,\varepsilon) = A \tan\left(\frac{A}{2\varepsilon}(\tfrac{1}{2}-t)\right) \quad \text{with} \quad A \tan\frac{A}{4\varepsilon} = 1 \ .$$

Here as $\varepsilon$ becomes small $x$ converges to the degenerate solution $u = 0$, $0 < t < 1$; but in this case we have "corner layers" at both ends of the interval as shown in figure 3.

figure 3/

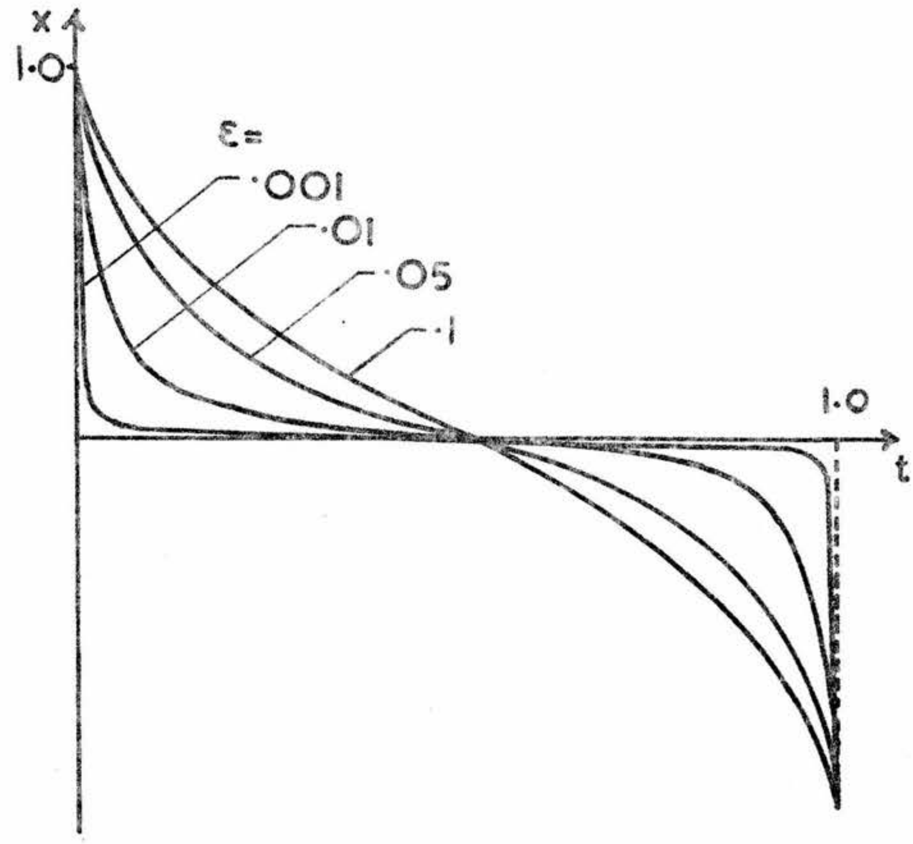


figure 3

Thus even in a simple problem a variety of phenomena occur. We notice that in each case discussed above $x(t,\varepsilon)$ has an essential singularity in $\varepsilon$ at $\varepsilon = 0$. This is a characteristic feature of such singular perturbation problems.

Problems of the type $P_\varepsilon$ have been discussed by various authors under varying assumptions. Coddington and Levinson [2] considered problems in which

(1.8) $$F(t,x,x',\varepsilon) = f(t,x)x' + g(t,x)$$

(1.9) and $x(0,\varepsilon) = \alpha$, $x(1,\varepsilon) = \beta$ with $\alpha, \beta$ independent of $\varepsilon$.

They assumed that the reduced equation

$$f(t,u)u' + g(t,u) = 0$$

has a solution on $[0,1]$ with $u(1) = \beta$, that $f(t,x) \geq k > 0$, and that $f$ and $g$ are of class $C^1$ in a region of the $(t,x)$-plane which includes the graph of the degenerate problem $(t,u(t))$ and the point $(0,\alpha)$. They proved that (1.0) with (1.8) has a unique solution

x/

$x(t,\varepsilon)$ satisfying (1.9) and that $x(t,\varepsilon) \to u(t)$, $x'(t,\varepsilon) \to u'(t)$ as $\varepsilon \to 0$ uniformly on any subinterval $0 < \delta \leq t \leq 1$ .

In a later paper Haber and Levinson [11] obtained similar results for the problem

$$(1.10) \qquad \varepsilon x'' + F(t,x,x',\varepsilon) = 0 , \qquad 0 \leq t \leq 1$$

$$(1.11) \quad \text{with} \quad x(0,\varepsilon) = \alpha , \quad x(1,\varepsilon) = \beta$$

where $\alpha$, $\beta$ are independent of $\varepsilon$ and the degenerate equation admits a solution of the form

$$\begin{aligned} u &= g(t) & 0 \leq t \leq t_0 \\ &= h(t) & t_0 \leq t \leq 1 \quad , \end{aligned}$$

with $g(0) = \alpha$, $h(1) = \beta$ , $g(t_0) = h(t_0)$ but $g'(t_0) < h'(t_0)$ . Here it is assumed that $F$, $F_x$ and $F_{x'}$ are of class $C^1$ in a suitable region of $(t,x,x',\varepsilon)$ - space, and that

$$\begin{aligned} &F_{x'}(t,g(t),\ g'(t),\ 0) < 0 & 0 \leq t \leq t_0, \\ &F_{x'}(t,h(t),\ h'(t),\ 0) > 0 & t_0 \leq t \leq 1, \end{aligned}$$

$$\text{and} \quad F(t_0,u_0,y,0) < 0 \quad u_0 = u(t_0),\ g'(t_0) < y < h'(t_0) .$$

Wasow [20] investigated a slightly more general form of the problem considered by Coddington and Levinson, allowing $f$ and $g$ (see 1.8) to depend on $\varepsilon$ . Under the additional assumptions that $f$ and $g$ are analytic in $x$ and $\varepsilon$ and of class $C^2$ in $t$, he gave a constructional method for the representation of the solution in terms of convergent and asymptotic series. More precisely Wasow showed that $x(t,\varepsilon)$ possesses a convergent expansion in powers of $e^{-\phi(t)/\varepsilon}$, with $\phi(t) = \int_0^t f(s,u(s),0)ds$, whose coefficients are functions of $t$ and $\varepsilon$ having asymptotic expansions in powers of $\varepsilon$ . Co-incidentally his work furnished a new existence proof. An important aspect of this work was that it provided a clearer view of the analytical nature of $x(t,\varepsilon)$ and this was pursued further by Erdélyi [7,8]. Erdélyi was able to relax the differentiability and analyticity conditions, and more importantly he was able to replace the condition that $F$ be

a/

a linear function of $x'$ by the condition $\frac{1}{\varepsilon} F_{x'x'}$ be bounded. Considering the problem (1.12),(1.13), allowing $\alpha$ and $\beta$ to depend on $\varepsilon$, Erdélyi proved that if the degenerate equation

$$F(t,u,u',0) = 0$$

possesses a solution $u$ such that $u(1) = \beta(0)$, and if $|\alpha(0) - u(0)| < \mu_0$ where $\mu_0$ is a bound independent of $\varepsilon$, then for all sufficiently small $\varepsilon > 0$ (1.10), (1.11) possesses a solution $x(t,\varepsilon)$, and for this solution

$$x = u + O(\varepsilon) + O(e^{-\phi(t)/\varepsilon})$$
$$x' = u' + O(\varepsilon) + O(\tfrac{1}{\varepsilon} e^{-\phi(t)/\varepsilon})$$

hold uniformly for $0 \leqslant t \leqslant 1$. Here

$$\phi(t) = \int_0^t F_{x'}(s,u(s), u'(s), 0)ds$$

where $F_{x'}(t,u(t), u'(t),0)$ is assumed to be non-zero and positive on $[0,1]$, so that $\phi(t)$ is an increasing function of $t$. (Ensurihg the occurrence of a boundary layer at $t = 0$).

Subsequently Harris [12], Macki [17] and Willett [21] replaced the single second order equation (1.10) by a system of first order equations and studied more general boundary conditions than (1.11). In particular Macki discussed the system

(1.12) $$\frac{dx}{dt} = f(t,x,y,\varepsilon),$$
$$\varepsilon\frac{dy}{dt} = g(t,x,y,\varepsilon),$$

(1.13) with $a_1x(0,\varepsilon) + a_2y(0,\varepsilon) = \alpha(\varepsilon)$, $b_1x(1,\varepsilon) + b_2y(1,\varepsilon) = \beta(\varepsilon)$.

When this problem is specialised to that considered by Erdélyi, by setting $a_2 = b_2 = 0$, $a_1 = b_1 = 1$ and $f(t,x,y,\varepsilon) = y$, Macki's assumptions are more restrictive than those of [8]. If we simply set $f(t,x,y,\varepsilon) = y$, so that we are effectively considering (1.10) subject to (1.13) with $y = x'$, Macki's results give the estimates

$$x(t,\varepsilon) = u(t) + O(\varepsilon) + O(\varepsilon^{m+1}q),$$
$$x'(t,\varepsilon) = u'(t) + O(\varepsilon) + O(\varepsilon^{m}e^{-t/\varepsilon}),$$

for $0 \leqslant t \leqslant 1$, where $u(t)$ is the solution of the degenerate equation satisfying the condition $b_1u(1) + b_2u'(1) = \beta(0)$

and/

and $m$ and $q$ are defined by

$$m = 0 \quad \text{if} \quad a_2 \neq 0,$$
$$= -1 \quad \text{if} \quad a_2 = 0,$$
$$q = q(b_2, t, \varepsilon) = e^{-t/\varepsilon} + \frac{1}{\varepsilon} e^{-1/\varepsilon} \quad \text{if} \quad b_2 \neq 0,$$
$$= e^{-t/\varepsilon} \qquad \text{if} \quad b_2 = 0 .$$

More recently Willett [22] reconsidered the problem (1.10), (1.11) from a new standpoint, and obtained results which make it possible, under certain conditions, to assess the validity of a tentative approximation $u(t,\varepsilon)$ to its solution. Subsequently Erdélyi [9] clarified and extended Willett's results. He showed that if $u(t,\varepsilon)$ is an approximate solution of the boundary value problem (1.12), (1.13) in the sense that

$$\alpha(\varepsilon) - u(0,\varepsilon) = O(\eta) \ , \ \beta(\varepsilon) - u(1,\varepsilon) = O(\eta)$$

and

$$\varepsilon u'' + F(t,u,u',\varepsilon) = H(t,\varepsilon) + H^*(t,\varepsilon)$$

with $H = O(\eta)$ , $H^* = O(\varepsilon^{-1} \eta \, e^{-m\phi/\varepsilon})$, $m \geqslant 1$ and independent of $t$ and $\varepsilon$ , $\eta = \eta(\varepsilon) = O(\varepsilon)$ and

$$\phi = \phi(t\,\varepsilon) = \int_0^t \phi'(s,\varepsilon)ds, \quad \phi'(t,\varepsilon) = F_{x'}(t,u(t,\varepsilon),u'(t,\varepsilon),\varepsilon) + O(\varepsilon) \ ,$$

then, for all sufficiently small $\varepsilon > 0$, the problem (1.10),(1.11) has a solution $x$ such that

$$x - u = O(\eta), \quad x' - u' = O(\eta\Phi) \ ,$$

with $$\Phi = \Phi(t,\varepsilon) = 1 + \frac{1}{\varepsilon} e^{-\phi(t,\varepsilon)/\varepsilon} \qquad \text{if} \quad m > 1,$$
$$= 1 + \frac{1}{\varepsilon}\left(1 + \frac{\phi(t,\varepsilon)}{\varepsilon}\right) e^{-\phi(t,\varepsilon)/\varepsilon} \quad \text{if} \quad m = 1 .$$

The theorem stated and proved in later chapters is a natural development of [9] to cover the problem $P_\varepsilon$ .

Results of the nature outlined above enable us to consider separately methods for constructing approximate solutions. Several methods have been proposed for obtaining uniform asymptotic expansions for the solutions of singular perturbation problems under varying degrees of generality and rigour. As we cannot expect to obtain a uniform asymptotic expansion in the classical (Poincaré) sense, we employ 'general asymptotic expansions'. We use an asymptotic sequence $\{\mu_n(\varepsilon)\}$ , often/

often $\{\varepsilon^n\}$, as a 'scale' or 'gauge', and we say that a function $F(\varepsilon)$ has an asymptotic expansion with respect to this scale if $F(\varepsilon) - \sum_{n\leq N} F_n(\varepsilon) = o(\mu_N(\varepsilon))$ for each $N$. Two types of methods for obtaining such expansions have been identified by Cole [4], one he terms 'limit process expansions', the other 'two-variable expansions'. The method of matched expansions developed by Kaplun and Lagerstrom [13], Erdélyi [6], Van Dyke [19], Fraenkel [10] and others is typical of the former, and the methods developed by Cole and Kevorkian [3], Cochran [1], O'Malley [18] and others are typical of the latter. We will give here a brief description of both techniques using the example (1.5), (1.6) with $x_0 = \alpha$, $x_1 = \beta$, as an illustration. In chapter 5 we will apply the two-variable technique to an example of the type $P_\varepsilon$, a new application of the method.

Suppose the solution of (1.5), (1.6) has an asymptotic expansion $u(t,\varepsilon)$ of the form

$$u(t,\varepsilon) = \sum_{n=0}^{\infty} u_n(t)\varepsilon^n \quad \text{with} \quad u(1,\varepsilon) = x_1 = \beta .$$

Further suppose this 'outer expansion' is valid on $0 < \delta \leq x \leq 1$, with $\delta$ such that $\varepsilon/\delta \to 0$ as $\varepsilon \to 0+$. Formal substitution of $u$ into the differential equation (1.5) for $x$ yields

$$\sum u_n'' \varepsilon^{n+1} + (\sum u_n \varepsilon^n)(\sum u_n' \varepsilon^n) = 0 ,$$

and equating to zero the coefficients of powers of $\varepsilon$ we obtain

$$u_0 u_0' = 0$$

$$u_0'' + u_0 u_1' + u_1 u_0' = 0 ,$$

$$u_1'' + u_0 u_2' + u_2 u_0' + u_1 u_1' = 0 ,$$

etc., with $u_0(1) = \beta$ and $u_n(1) = 0$, $n \geq 1$, to meet the condition $u(1,\varepsilon) = \beta$. Integrating we obtain

$$u_0(t) = \beta , \quad u_n(t) = 0, \quad n \geq 1 ,$$

and our outer solution is simply

$$u(t,\varepsilon) = \beta .$$

Next we obtain an inner expansion which will be valid in the boundary layer region and will satisfy the condition at $t = 0$.

To/

To do this we follow Prandtl's idea and use a 'stretching transformation' $\tau = \tau(t,\varepsilon)$ which transforms the differential equation into one in which the higher-order derivatives dominate. A suitable stretching in this case is $\tau = t/\varepsilon$, so that the differential equation becomes

$$\ddot{x} + x\dot{x} = 0$$

where $(\dot{\ })$ denotes $\frac{d}{d\tau}(\ )$. We now suppose that $x$ has an "inner expansion" $w(\tau,\varepsilon)$ of the form

$$w(\tau,\varepsilon) = \sum_{n=0}^{\infty} w_n(\tau)\varepsilon^n$$

with $w(0,\varepsilon) = \alpha$, and that this expansion is valid for $t = \varepsilon\tau = O(1)$. Substitution into the differential equatioh followed by the equation of coefficients of powers of to zero yields

$$\ddot{w}_0 + w_0\dot{w}_0 = 0$$
$$\ddot{w}_1 + w_0\dot{w}_1 + \dot{w}_0 w_1 = 0$$
$$\ddot{w}_2 + w_0\dot{w}_2 + \dot{w}_0 w_2 + w_1\dot{w}_1 = 0,$$

etc., with $w_0(0) = \alpha$, $w_n(0) = 0$, $n \geqslant 1$, to meet the condition $w(0,\varepsilon) = \alpha$. Integrating these equations and using the conditions imposed at $t = 0$ gives

$$w_0 = A_0 \tanh\left(\frac{A_0\tau}{2} + B_0\right), \quad \text{with } B_0 = \tanh^{-1}\frac{\alpha}{A_0},$$

$$w_1 = A_1\left[\frac{1}{A_0}\tanh\left(\frac{A_0\tau}{2} + B_0\right) + \left(\tfrac{1}{2}\tau - \frac{\alpha}{A_0^2 - \alpha^2}\right)\operatorname{sech}^2\left(\frac{A_0\tau}{2} + B_0\right)\right]$$

etc.. Each term in our inner expansion contains one undetermined constant of integration. This constant is fixed by 'matching' our inner and outer expansions. If the domains of validity of the inner and outer expansions 'overlap', in the domain of overlap the difference $|u(t,\varepsilon) - w(t,\varepsilon)|$ tends to zero as $\varepsilon \to 0$, thus determining the unknown constants in the inner solution. This may be formalised with the aid of an "intermediate limit". We set $t = \eta T$ where $\eta(\varepsilon) \to 0$ while $\frac{\varepsilon}{\eta} \to 0$, $\varepsilon \leqslant \eta \leqslant 1$, $\varepsilon \to 0$, and define the $\eta$-limit of a function $a(t,\varepsilon)$ as $\lim_\eta a(t,\varepsilon) = \lim_{\varepsilon \to 0_+} a(t,\varepsilon)$ with $T$ fixed; such a limit is called an intermediate limit. Then in the domain of overlap

lim/

$\lim_{\eta} | u-w | = 0$ . Applying this to $u$ and $w$ obtained above we find

$$A_o = \beta \text{ and } A_n = 0 , \quad n > 1 .$$

Thus $\quad w_o = \beta \tanh(\frac{\beta\tau}{2} + \gamma)$ , where $\gamma = \tanh^{-1} \frac{\alpha}{\beta}$ ,

and $\quad w_n = 0 \quad n \geqslant 1$ .

Having completely determined both inner and outer expansions, we construct a composite expansion which we hope will be uniformly valid on $0 \leqslant t \leqslant 1$ . We do this by adding together the inner and outer expansions and subtracting their common part c.p.(u,w), i.e.

$$x(t,\varepsilon) \sim u + w - \text{c.p.}(u,w) .$$

We may obtain the common part of $u$ and $w$ either by setting $t = \varepsilon\tau$ in the outer expansion or by setting $\tau = t/\varepsilon$ in the inner expansion. In our example this process is particularly simple since $u(\varepsilon\tau) = \beta$ , i.e. $\text{c.p.}(u,w) = \beta$ and $x(t,\varepsilon) \sim \beta \tanh(\frac{\beta t}{2\varepsilon} + \gamma)$ .

The fundamental concepts of matched asymptotic expansions have been re-examined in depth in recent papers by Lagerstrom [16] and Eckhaus [5]. Because the technique sketched out above is difficult and laborious in some applications, simplifying modifications have been proposed. In particular the 'asymptotic matching principle' proposed by Van Dyke [19] eliminates the use of intermediate limits. Its validity has been discussed in detail by Fraenkel [10].

We turn now to the second technique for the construction of approximate solutions, the two-variable method. It is difficult to ascertain the precise origin of the idea of introducing two independent variables in the development of an approximate solution. Kevorkian [14] suggests that Cole originally proposed the idea of explicitly using two independent variables but he does point out that the idea is implicit in the much earlier 'method of averaging' developed by Krylov and Bogoliubov [15].* Kevorkian and Cole developed the method for various initial value problems. Cochran [1] also developed a two-variable/

*Indeed this goes back to Poincaré's original work.

two-variable method and applied it to various problems including the boundary value problems

$$\varepsilon x'' + xx' - x = 0, \ 0 \leq t \leq 1, \ \ x(0) = \alpha, \ x(1) = \beta ,$$

and

$$\varepsilon x'' - (\tfrac{1}{2} - t)x' - x = 0, \ \ 0 \leq t \leq 1, \ x(0) = \alpha, \ x(1) = \beta .$$

We will apply the technique to the problem considered before. We pose

$$x(t,\varepsilon) \sim \sum_{n=0}^{\infty} x_n(t,\tau)\,\varepsilon^n$$

where $\tau = \tau(t,\varepsilon)$ is a boundary layer coordinate and in this case $\tau = t/\varepsilon$ is a suitable choice. The formal substitution of the series for $x$ into the differential equation (1.5) yields

$$\sum (x_{ntt} + \frac{2}{\varepsilon} x_{nt\tau} + \frac{1}{\varepsilon^2} x_{n\tau\tau})^{\,n+1} + \sum x_n \varepsilon^n \sum (x_{nt} + \frac{1}{\varepsilon} x_{n\tau})\,\varepsilon^n = 0$$

where the subscripts $t, \tau$ denote partial differentiation. Equating the coefficients of powers of $\varepsilon$ to zero gives

$$x_{o\tau\tau} + x_o x_{o\tau} = 0 ,$$

$$x_{1\tau\tau} + (x_o x_1)_\tau + 2x_{ot\tau} + x_o x_{ot} = 0 ,$$

$$x_{2\tau\tau} + (x_o x_2)_\tau + 2x_{1t\tau} + x_{ott} + x_1 x_{1\tau} + x_o x_{1t} + x_1 x_{ot} = 0 ,$$

etc.. To meet the boundary conditions at $t = 0$ and $t = 1$, we impose the conditions $x_o(0,0) = \alpha$ , $x_o(1,\infty) = \beta$, $x_n(0,0) = 0$, $x_n(1,\infty) = 0$, $n \geq 1$ . Integrating the first of these partial differential equations we obtain

$$x_o(t,\tau) = u_o(t) \ \tanh(\tfrac{1}{2} u_o(t)\tau + v_o(t) )$$

where $u_o(t)$, $v_o(t)$ are arbitrary functions of $t$ arising from the two integrations with respect to $\tau$ . We shall see below that $u_o(t)$ is the first term of the outer expansion obtained before. Substitution into the differential equation for $x_1$ followed by integration yields

$$x_1(t,\tau) = u_o\{-\tfrac{1}{4} u_o' \tau^2 - \tfrac{1}{2} v_o' + \tfrac{1}{2}u_1\tau + v_1 + \frac{u_o'}{u_o^2}\} \ \mathrm{sech}^2(\tfrac{1}{2}u_o\tau + v_o)$$

$$- \frac{2u_o'}{u_o} + \{- u_o'\tau + v_o' + u_1\} \ \tanh (\tfrac{1}{2} u_o\tau + v_o ) .$$

Since we require $x_1(t,\tau)$ to be a bounded function of $\tau$ , the arbitrary/

arbitrary function $u_o(t)$ must be such that $u_o' = 0$. We also require $x_o(1,\infty) = \beta$ which implies $u_o(1) = \beta$. Thus $u_o$ is completely determined $u_o = \beta$, $0 < t < 1$. $\tau$ is essentially a boundary layer coordinate so that we would expect $x(t,\varepsilon)$ to depend on $-\tau$ exponentially. Indeed we would expect

$$\sum_{n=o}^{\infty} x_n(t,\tau)\varepsilon^n \to \sum_{n=o}^{\infty} u_n(t)\varepsilon^n \quad \text{(the outer expansion)}$$

and
$$\frac{\partial}{\partial\tau}\left(\sum_{n=o}^{\infty} x_n(t,\tau)\varepsilon^n\right) \to 0$$

as $\tau \to \infty$. These imply that

$$x_n(t,\tau) \to u_n(t) \quad \text{and} \quad x_{n\tau}(t,\tau) \to 0$$

as $\tau \to \infty$. Inspecting the expression for $x_1(t,\tau)$ above we deduce that $v_o'(t) = 0$ and hence $v_o(t) = \gamma$ a constant such that $x_o(0,0) = \alpha$. Thus $x_o(t,\tau)$ is completely determined

$$x_o = \beta \tanh\left(\tfrac{1}{2}\beta\tau + \gamma\right) \text{ with } \gamma = \tanh^{-1}\frac{\alpha}{\beta},$$

while the expression for $x_1$ becomes

$$x_1 = \beta\left(\tfrac{1}{2}u_1\tau + v_1\right)\operatorname{sech}^2\left(\tfrac{1}{2}\beta\tau + \gamma\right) + u_1 \tanh\left(\tfrac{1}{2}\beta\tau + \gamma\right).$$

The arbitrary functions $u_1(t)$, $v_1(t)$ are determined at the next step in the construction of the approximation in the same way as $u_o$, $v_o$ were found above. We find then that $u_1' = 0$ and $v_1' = 0$, and using the boundary conditions we obtain $u_1(t) = 0$ and $v_1(t) = 0$, so that $x_1(t,\tau) = 0$. Indeed all subsequent terms are identically zero. Thus we obtain again the approximation

$$x(t,\varepsilon) \sim \beta \tanh\left(\frac{\beta t}{2\varepsilon} + \gamma\right).$$

## Chapter 2

## The Linear Differential Equation

In the proof of the result stated in the next chapter we will regard solutions of

(2.0) $\qquad \varepsilon x'' + F(t,x,x',\varepsilon) = 0$

as non-linear perturbations of suitable linear differential equations. We list here some standard results concerning the solutions of linear equations and derive some specialised results for use in chapter 4. Order symbols, such as $O(\varepsilon)$, $o(1)$, involve $\varepsilon \to 0+$, and will be understood to hold uniformly in $t$, $0 \leqslant t \leqslant 1$.

Consider the differential equation

(2.1) $\qquad \varepsilon V'' + p(t,\varepsilon)V' + q(t,\varepsilon)V = 0 , \quad 0 \leqslant t \leqslant 1 .$

We assume (i) $0 < \varepsilon < \varepsilon_0$ ;

(ii) $p(t,\varepsilon) = \phi'(t,\varepsilon) + \varepsilon p_1(t,\varepsilon)$ ;

(iii) $\phi(0,\varepsilon) = 0$, $\phi'(t,\varepsilon) > 0$ , $\phi$ twice continuously differentiable with respect to $t$, and $\phi'$, $1/\phi'$, $\phi'' = O(1)$ as $\varepsilon \to 0+$ ;

(iv) $p_1(t,\varepsilon)$, $q(t,\varepsilon)$ are continuous in $t$ for each fixed $\varepsilon$ and $p_1, q = O(1)$ as $\varepsilon \to 0+$ .

We set

(2.2) $\quad \psi(t) = \psi(t,\varepsilon) = \int_0^t q(s,\varepsilon)/\phi'(s,\varepsilon)\, ds$ ;

(2.3) $\quad \chi(t) = \chi(t,\varepsilon) = \int_1^t (q(s,\varepsilon) - p_1(s,\varepsilon))/\phi'(s,\varepsilon)\, ds$ ;

(2.4) $\quad \theta(s,t) = (\phi(s,\varepsilon) - \phi(t,\varepsilon))/\varepsilon$ .

We note that $\psi, \psi' = O(1)$ , and $\operatorname{sgn} \theta(s,t) = \operatorname{sgn}(s-t)$ .

Let $V_A$, $V_B$, $K(t,s)$ be those solutions of (2.1) such that

(2.5) $\quad V_A(0) = 1$ , $V_A'(0) = -\psi'(0)$ ;

(2.6) $\quad V_B(1) = -e^{-\phi(1)/\varepsilon}$ , $V_B'(1) = \phi'(1)e^{-\phi(1)/\varepsilon}$ ;

(2.7) $\quad K(s,s) = 0$ , $\varepsilon K_t(s,s) = 1$ .

Then/

Then usihg the methods outlined in [9] we obtain

(2.8) $\quad V_A(t) = e^{-\psi(t)} + O(\varepsilon),\ V_A'(t) = -\psi'(t)e^{-\psi(t)} + O(\varepsilon)$ ;

(2.9) $\quad V_B(t) = \varepsilon e^{-\phi(t)/\varepsilon}\{e^{\chi(t)} + O(\varepsilon)\},\ V_B'(t) = \phi'(t)e^{-\phi(t)/\varepsilon}\{e^{\chi(t)} + O(\varepsilon)\}$ ;

(2.10) $$K(t,s) = \frac{V_A(s)\,V_B(t) - V_A(t)\,V_B(s)}{\varepsilon\{V_A(s)V_B'(s) - V_A'(s)V_B(s)\}},$$

$$= -\frac{1}{\phi'(s)}\ e^{\chi(t)-\chi(s)+\theta(s,t)}(1+O(\varepsilon)) + \frac{1}{\phi'(s)}\ e^{\psi(s)-\psi(t)} + O(\varepsilon),$$

$$K_t(t,s) = \frac{\phi'(t)}{\varepsilon\phi'(s)}\ e^{\chi(t)-\chi(s)+\theta(s,t)}(1+O(\varepsilon)) - \frac{\psi'(t)}{\phi'(s)}\ e^{\psi(s)-\psi(t)} + o(1).$$

Let $V_1(t,\varepsilon)$, $V_2(t,\varepsilon)$ be the solutions of the homogeneous equation (2.1) satisfying

(2.11) $\quad V_1'(0,\varepsilon) = 0\ ,\quad b_1V_1(1,\varepsilon) + b_2V_1'(1,\varepsilon) = 1$ ;

(2.12) $\quad a_1V_2(0,\varepsilon) + \varepsilon\, a_2V_2'(0,\varepsilon) = 1,\quad b_1V_2(1,\varepsilon) + b_2V_2'(1,\varepsilon) = 0$ .

We may write $V_1$, $V_2$ in terms of $V_A$, $V_B$

$$V_1 = A_1V_A + B_1V_B\ ,\quad V_2 = A_2V_A + B_2V_B\ ,$$

where
$$A_1 = -V_B'(0,\varepsilon)/\Delta_1\ ,\quad B_1 = V_A'(0,\varepsilon)/\Delta_1\ ,$$

$$\Delta_1 = V_A'(0)(b_1V_B(1) + b_2V_B'(1)) - V_2'(0)(b_1V_A(1) + b_2V_A'(1))\ ,$$

$$A_2 = (b_1V_B(1) + b_2V_B'(1))/\Delta_2,\quad B_2 = -(b_1V_A(1) + b_2V_A'(1))/\Delta_2,$$

$$\Delta_2 = (a_1V_A(0) + \varepsilon\, a_2V_A'(0))(b_1V_B(1) + b_2V_B'(1)) -$$

$$(b_1V_A(1) + b_2V_A'(1))\,(a_1V_B(0) + \varepsilon\, a_2V_B'(0))\ .$$

Under the additional assumptions

(v) $\quad a_1,\ a_2,\ b_1,\ b_2 = O(1)$ ;

(vi) $\quad [a_1 - a_2\,\phi'(0,\varepsilon)]^{-1} = O(1)$ ;

(vii) $\quad [b_1 - b_2q(1,\varepsilon)/\phi'(1,\varepsilon)]^{-1} = O(1)$ ;

we obtain

$$V_1(t,\varepsilon)\ ,\quad V_1'(t,\varepsilon) = O(1)\ ,\quad \text{and}$$

(2.13) $\quad V_2(t,\varepsilon) = O(e^{-\phi(t)/\varepsilon}) + O(\frac{1}{\varepsilon}\,e^{-\phi(1)/\varepsilon}),\ V_2'(t,\varepsilon) = O(\frac{1}{\varepsilon}e^{-\phi(t)/\varepsilon})$ .

Further let us denote by $T_{1\lambda}f$ and $T_{2\mu}f$ those solutions of the inhomogeneous differential equation

(2.14) $\quad \varepsilon V'' + pV' + qV = f$

satisfying/

satisfying

$$(2.15)\qquad (T_1f)'(0) = 0\ ,\quad b_1(T_1f)(1) + b_2(T_1f)'(1) = \lambda\ ;$$

$$(2.16)\qquad a_1(T_2f)(0) + \varepsilon a_2(T_2f)'(0) = \mu\ ,\ b_1(T_2f)(1) + b_2(T_2f)'(1) = 0\ .$$

Here $f, \lambda, \mu$ may depend on $\varepsilon$. Explicitly

$$(2.17)\quad (T_1f)(t) = [\lambda - \int_0^1 \{b_1K(1,s)+b_2K_t(1,s)\} f(s)ds]V_1(t) + \int_0^t K(t,s)f(s)ds;$$

$$(2.18)\quad (T_2f)(t) = [\mu + \int_0^1 \{a_1K(0,s)+\varepsilon a_2K_t(0,s)\} f(s)ds]V_2(t) - \int_0^1 K(t,s)f(s)ds.$$

From (2.10) we note that:

$$\int_0^t K(t,s)\ O(1)ds,\quad \int_0^t K_t(t,s)\ O(1)ds = O(1)\ ,$$

and $\int_t^1 K(t,s)O(e^{-m\phi(s)/\varepsilon})ds,\ \varepsilon\int_t^1 K_t(t,s)O(e^{-m\phi(s)/\varepsilon})ds$

$$= O(\varepsilon e^{-\phi(t)/\varepsilon})\quad \text{if}\quad m > 1\ ,$$

$$= O(e^{-\phi(t)/\varepsilon})\quad \text{if}\quad m = 1\ .$$

Using these results and assuming that (v), (vi) and (vii) hold, we obtain for any $\eta = \eta(\varepsilon)$

$$(2.19)\quad T_{1\lambda}\ O(\eta)\ ,\ (T_{1\lambda}\ O(\eta))' = O(\lambda) + O(\eta)\ ,$$

$$(2.20)\quad T_{2\mu}O(\eta e^{-m\phi/\varepsilon}) = O(\mu(e^{-\phi/\varepsilon}+\tfrac{1}{\varepsilon}e^{-\phi(1)/\varepsilon})) + O(\varepsilon\eta(e^{-\phi/\varepsilon}+\tfrac{1}{\varepsilon}e^{-\phi(1)/\varepsilon})),$$

$$\varepsilon(T_{2\mu}O(\eta e^{-m\phi/\varepsilon}))' = O(\mu e^{-\phi/\varepsilon}) + O(\varepsilon\eta e^{-\phi/\varepsilon})\quad \text{if}\quad m > 1\ ,$$

and

$$(2.21)\quad T_{2\mu}O(\eta e^{-\phi/\varepsilon}) = O(\mu(e^{-\phi/\varepsilon}+\tfrac{1}{\varepsilon}e^{-\phi(1)/\varepsilon})) + O(\eta(e^{-\phi/\varepsilon}+\tfrac{1}{\varepsilon}e^{-\phi(1)/\varepsilon}))\ ,$$

$$\varepsilon(T_{2\mu}O(\eta e^{-\phi/\varepsilon}))' = O(\mu e^{-\phi/\varepsilon}) + O(\varepsilon\eta e^{-\phi/\varepsilon})\ .$$

We shall also need a finer estimate than (2.21) having the form (2.20). It is

$$(2.22)\quad T_{2\mu}O(\eta e^{-\phi/\varepsilon}) = O(\mu(e^{-\phi/\varepsilon}+\tfrac{1}{\varepsilon}e^{-\phi(1)/\varepsilon})) + O(\eta((\varepsilon+\phi)e^{-\phi/\varepsilon}+\tfrac{e}{\varepsilon}^{-\phi(1)/\varepsilon}))\ ,$$

$$(2.23)\quad (T_{2\mu}O(\eta e^{-\phi/\varepsilon}))' = O(\mu(e^{-\phi/\varepsilon})) + O(\eta(\varepsilon+\phi)e^{-\phi/\varepsilon}) + O(\tfrac{\eta}{\varepsilon}e^{-\phi(1)/\varepsilon})\ .$$

To obtain these results we re-write (2.18) as

$$(T_2f)(t) = [\mu - \int_0^1 \{a_1V_A(0)+\varepsilon a_2V_A'(0)\}\ V_B(s)f(s)W(s)\ ds\ ]\ V_2(t)$$

$$+A_2V_A(t)\int_0^1 \{a_1V_B(0) + \varepsilon a_2V_B'(0)\}\ V_A(s)\ W(s)f(s)\ ds$$

$$+B_2V_B(t)\int_0^1 \{a_1V_B(0) + a_2V_B'(0)\}V_A(s)\ W(s)f(s)ds - \int_0^1 V_A(s)V_B(t)W(s)f(s)ds$$

$$+\int_0^t V_A(s)V_B(t)\ W(s)\ f(s)\ ds + \int_t^1 V_A(t)\ V_B(s)\ W(s)f(s)\ ds\ ,$$

where/

where $W(t,\varepsilon) = \{\varepsilon(V_A(t)V_B'(t) - V_A'(t)V_B(t))\}^{-1} = O(\frac{1}{\varepsilon} e^{\phi(t)/\varepsilon})$ ,

Now $B_2 \{a_1 V_B(0) + \varepsilon a_2 V_B'(0)\} - 1 = -(a_1 V_A(0) + \varepsilon a_2 V_A'(0))\, A_2$ ,

where $A_2 = O(\frac{1}{\varepsilon} e^{-\phi(1)/\varepsilon})$ , and we deduce that

(2.24) $V_3(t,\varepsilon) = A_2[(a_1 V_B(0) + \varepsilon a_2 V_B'(0))V_A(t) - (a_1 V_A(0) + \varepsilon a_2 V_A'(0))V_B(t)]$
$= O(e^{-\phi(1)/\varepsilon})$ .

Thus $T_2 f = \mu V_2 - I_1 V_2 + I_2 V_3 + I_3 V_B + I_4 V_A$ , where

$I_1 = \int_0^1 \{a_1 V_A(0) + \varepsilon a_2 V_A'(0)\} V_B W f\, ds$ , $I_2 = \int_0^1 V_A W f\, ds$ , $I_3 = \int_0^t V_A W f\, ds$

and $I_4 = \int_t^1 V_B\, W f\, ds$ .

With $f = O(\eta\, e^{-\phi(t)/\varepsilon})$ , we obtain

$I_1 = O(\varepsilon\eta)$ , $I_2 = O(\frac{\eta}{\varepsilon})$ , $I_3 = O(\eta\phi/\varepsilon)$ and $I_4 = O(\eta\varepsilon\, e^{-\phi/\varepsilon})$ ,

and these relations together with (2.8), (2,9), (2.13) and (2.24) yield (2.22). (2.23) follows in a similar manner.

## Chapter 3

## The Non-linear Differential Equation: Assumptions and Main Result

We now formulate precisely the assumptions about $P_\varepsilon$ . $F(t,x,y,\varepsilon)$ will often be abbreviated to $F(x,y)$ and its partial derivatives will be denoted by subscripts. Order symbols such as $o(1)$ and $O(e^{-\phi(t,\varepsilon)/\varepsilon})$ will be understood to hold uniformly in $t$ for $0 \leq t \leq 1$ as $\varepsilon \to 0+$ . $u(t,\varepsilon)$ is an "approximate solution" of $P_\varepsilon$ in the sense that

$$(3.0) \qquad \varepsilon u'' + F(t,u,u',\varepsilon) = H(t,\varepsilon) + H^*(t,\varepsilon)$$

and (3.1) $\alpha(\varepsilon) - a_1 u(0,\varepsilon) - \varepsilon a_2 u'(0,\varepsilon) = O(\eta)$

$$\beta(\varepsilon) - b_1 u(1,\varepsilon) - b_2 u'(1,\varepsilon) = O(\eta)$$

where (3.2) $H = O(\eta)$, $H^* = O(\varepsilon^{-1}\eta\, e^{-m\phi/\varepsilon})$ , $\eta = \eta(\varepsilon)$ is a given (small) function of $\varepsilon$ , $\phi = \phi(t,\varepsilon) = \int_0^t \phi'(s,\varepsilon)ds$ with $\phi(t,\varepsilon)$ such that $F_y(t,u,u',\varepsilon) = \phi'(t,\varepsilon) + O(\varepsilon)$ , and $m$ is a parameter independent of both $t$ and $\varepsilon$. We will often refer to $u(t,\varepsilon)$ as an approximate solution of $P_\varepsilon$ and will abbreviate it to $u(t)$ or even $u$ .

The splitting of (3.0) into two parts may appear to be somewhat artificial, especially as the 'boundary layer error' term $H^*$ involves $\exp(-\phi/\varepsilon)$ where $\phi$ itself depends on $u$. However this type of approximation does occur in practice. For example, from Erdélyi's (1962) paper [8] and Macki's result [17] we would expect the crudest type of approximation to the solution of

$$\varepsilon x'' + x' + x^{N+1} = 0 \quad \text{subject to (1.1) and (1.2)} ,$$

to take the form

$$(3.3) \qquad u(t,\varepsilon) = u_o(t) + Ae^{-t/\varepsilon}$$

where $u_o(t)$ is the solution of the degenerate problem,

$$u_o' + u_o^{N+1} = 0, \quad b_1 u_o(1) + b_2 u_o'(1) = \beta(0) ,$$

and $A = (\alpha(0) - a_1 u_o(0))/(a_1 - a_2)$ . The substitution of $u$ into the left handside of the differential equation yields

$$(3.4) \qquad \varepsilon u'' + u' + u^{N+1} = O(\varepsilon) + O(e^{-t/\varepsilon}) .$$

In this example $F_y(u,\dot{u}) = 1$ , so that we may choose $\phi(t,\varepsilon) = t$ , and thus (3.4) may be written in the form (3.0) .

Not/

Not all first approximations will take the exact form (3.3). For example, a first approximation to the solution of

$$\varepsilon x'' + xx' - x = 0 \qquad \text{subject to (1.1) and (1.2)},$$

is, for certain values of $\alpha$ and $\beta$,

$$u(t) = u_o(t)\tanh(t\,u_o/(2\varepsilon) + c) \quad \text{where} \quad u_o(t) = t - 1 + (\beta - b_2)/b_1$$

is the solution of the degenerate problem and $c$ is a constant such that $a_1 u(o) + \varepsilon a_2 u'(o) = \alpha + 0(\varepsilon)$.

With this approximation

$$(3.5) \qquad \varepsilon u'' + uu' - u = 0(\varepsilon) + 0(e^{-tu_o/\varepsilon}).$$

In this example $F_y(u,u') = u(t,\varepsilon)$, so that we may choose $\phi'(t,\varepsilon) = u(t,\varepsilon)$. Since $u_o(t) \geq u(t)$ for $0 \leq t \leq 1$, $\int_o^t u(s,\varepsilon)ds \leq \int_o^t u_o(s)ds \leq t\,u_o(t)$ there, and we have that $e^{-tu_o/\varepsilon} \leq e^{-\phi/\varepsilon}$. Thus (3.5) may also be written in the standard form.

The problem $P_\varepsilon$ differs from that considered by Macki [17] not only in that the equivalent system of differential equations is slightly more simple than the one he considers (under more restrictive assumptions than ours), but also in that the boundary condition at $t = 0$ is different, $\varepsilon a_2 x'(0)$ replacing Macki's $a_2 y(0)$. While it is possible to obtain a similar result to the one presented here for the problem Macki poses (with $y = x'$), the result so obtained is considerably more complicated. Lastly while Macki assumes $a_1, a_2, b_1, b_2$ to be independent of $\varepsilon$, here we allow them to be bounded functions of $\varepsilon$.

We set (i) $p(t,\varepsilon) = F_y(u,u') = \phi'(t,\varepsilon) + \varepsilon p_1(t,\varepsilon)$,

(ii) $q(t,\varepsilon) = F_x(u,u')$,

(iii) $\Phi(t,\varepsilon) = 1 + \frac{1}{\varepsilon}e^{-\phi(t,\varepsilon)/\varepsilon}$ if $m > 1$,

$= 1 + \frac{1}{\varepsilon}\left(1 + \frac{\phi(t,\varepsilon)}{\varepsilon}\right)e^{-\phi(t,\varepsilon)/\varepsilon}$ if $m = 1$,

(iv) for a fixed $\delta > 0$,

$$\mathfrak{D}_\delta = \{(t,x,y,\varepsilon) \mid 0 \leq t \leq 1,\ |x-u(t)| < \delta,\ |y-u'(t)| < \delta\,\Phi(t,\varepsilon),\ 0 < \varepsilon < \varepsilon_o\}.$$

<u>Assumptions</u>/

Assumptions

(A) For each positive $\varepsilon < \varepsilon_0$, $u(t,\varepsilon)$ is a twice continuously differentiable function of $t$.

(B) For some positive $\delta$, $F(t,x,y,\varepsilon)$ is defined in $\mathfrak{D}_\delta$ and has there the partial derivatives $F_x$, $F_y$, $F_{xx}$, $F_{xy}$, $F_{yx}$ and $F_{yy}$. Further, for each fixed $\varepsilon$, $F$ and its first and second order partial derivatives are continuous functions of $t$, $x$ and $y$.

(C) (i) above holds with $p_1(t,\varepsilon) = O(1)$ and $\phi'(t,\varepsilon)$ a continuously differentiable positive function of $t$ such that $\phi'$, $(\phi')^{-1}$ and $\phi''$ are all bounded.

(D) $q(t,\varepsilon) = O(1)$.

(E) $F_{xx}$, $F_{xy}$ and $\varepsilon^{-1} F_{yy}$ are all bounded.

(F) $a_1$, $a_2$, $b_1$ and $b_2$ are all bounded, with

$$\{a_1 - a_2\,\phi'(0,\varepsilon)\}^{-1} = O(1)$$

and $$\{b_1 - b_2\,q(1,\varepsilon)/\phi'(1,\varepsilon)\}^{-1} = O(1)\ .$$

(G) (3.0), (3.1) and (3.2) hold with $m \geq 1$ and $\eta = \eta(\varepsilon) = O(\varepsilon)$.

Result

If (A) - (G) hold, then, for all sufficiently small $\varepsilon > 0$, the boundary value problem $P_\varepsilon$ possesses a solution $x$ such that $x-u = O(\eta)$ and $x'-u' = O(\eta\bar{\Phi})$.

Both the assumptions (excepting (F)) and the result resemble those of Erdélyi [9] for the differential equation (1.0) subject to the boundary conditions $x(0,\varepsilon) = \alpha$, $x(1,\varepsilon) = \beta$. The restrictions placed on the values of $a_1$, $a_2$, $b_1$ and $b_2$ in assumption (F) merit some discussion. These restrictions are made necessary by the method used to establish the result; but examples exist which justify such restrictions. Consider the simple problem of $\varepsilon x'' + x' = 0$ subject to (1.1) and (1.2) where $a_1$, $a_2$, $b_1$ and $b_2$ are independent of $\varepsilon$. Here $F_y(x,y) = 1$ and $F_x(x,y) = 0$, so that $q(t,\varepsilon) \equiv 0$ and we may choose $\phi'(t,\varepsilon) \equiv 1$ for all $u(t,\varepsilon)$. The restrictions in this case reduce to/

to $a_1 \neq a_2$ and $b_1 \neq 0$. If either of these conditions is broken there is no bounded solution to the problem; indeed when $b_1 = 0$ with $a_1 = 0$ there is no solution at all. Another example is furnished in chapter 5 where such restrictions are necessary for the construction of an approximate solution to a boundary value problem.

# Chapter 4

## The Non-linear Differential Equation: Proof of Main Result

The approach to the problem of proof follows an idea developed and used by Wasow. We first construct a solution $x^*(t,\varepsilon) = u(t,\varepsilon) + v(t,\varepsilon)$ which satisfies (1.0) and the boundary condition at $t = 1$ (1.2). The functions $v$ and $v'$ are $O(\eta)$ uniformly on $0 \leq t \leq 1$, so that $x^*$ is uniformly close to $u$ there. $v$ is usually termed the outer correction. We then add a correction so that the resulting function $x^*(t,\varepsilon) + w(t,\varepsilon)$ is a solution to $P_\varepsilon$. $w(t,\varepsilon)$ is the inner (or boundary layer) correction, its name reflecting the fact that it is the dominant correction term close to $t = 0$ and is negligible elsewhere. Instead of using power series methods as Wasow did, we follow Erdélyi in using constructions based on integral equation methods. $v$ is constructed by 'linearising' the differential equation (1,0) around $u$, and $w$ by 'linearisation' about $x^* = u + v$.

### The Outer Correction

We set $x^* = u + v$ such that

(4.0) $\quad \varepsilon x^{*''} + F(t,x^*,x^{*'},\varepsilon) = H^*(t,\varepsilon)$

with (4.1) $x^{*'}(0) = u'(0)$ and $b_1 x^*(1) + b_2 x^{*'}(1) = \beta(\varepsilon)$ .

Then $v$ satisfies the non-linear equation

(4.2) $\quad \varepsilon v'' + pv' + qv = G(t,v,v',\varepsilon)$

and the boundary conditions

(4.3) $\quad v'(0) = 0$, and $b_1 v(1) + b_2 v'(1) = \beta(\varepsilon) - b_1 u(1) - b_2 u'(1) = \lambda(\varepsilon) = O(\eta)$ ,

where $\quad p = p(t,\varepsilon) = F_y(u,u') = \phi'(t,\varepsilon) + \varepsilon p_1(t,\varepsilon)$

and $\quad q = q(t,\varepsilon) = F_x(u,u')$, as in assumption C,

and $\quad G(t,v,v',\varepsilon) = G(v,v') = -H(t,\varepsilon) - \{F(u+v,u'+v') - F(u,u') - pv'-qv\}$ .

By assumptions (C) and (D), p and q satisfy the conditions of chapter 2, so that we may rewrite (4.2) as the functional equation

(4.4) $$v = T_1 G ,$$

with/

with $\lambda$ given in (4.3). We construct $v$ by successive approximations, through a sequence $v_n$ with $v_{-1} \equiv 0$, $v_n = T_1 G_{n-1}$ for $n = 0,1,2,\ldots,$ where $G_n = G_n(t,\varepsilon)$ is written for $G(t, v_n(t,\varepsilon), v_n'(t,\varepsilon),\varepsilon)$.

Since $\lambda = O(\eta)$ and $G_{-1} = G(t,0,0,\varepsilon) = -H(t,\varepsilon) = O(\eta)$, it follows from (2.19) that both $v_o$ and $v_o'$ are $O(\eta)$, i.e.

(4.5) $|v_o(t,\varepsilon)|, |v_o'(t,\varepsilon)| \leq D_1 \eta$ for some $D_1$ independent of $t$ and $\varepsilon$.

It also follows from (2.19) (with $\lambda = 0$), that if

$$|f(t,\varepsilon) - g(t,\varepsilon)| \leq C(\varepsilon),$$

then (4.6) $|T_1 f - T_1 g|, \quad |(T_1 f - T_1 g)'| \leq D_2 C(\varepsilon)$.

By the Mean Value Theorem

$$\begin{aligned} G_r - G_{r-1} &= F(u+v_r, u'+v_r') - F(u+v_{r-1}, u'+v_{r-1}') - F_y(u,u')(v_r'-v_{r-1}') \\ &\qquad - F_x(u,u')(v_r - v_{r-1}) \\ &= [F_x(u+v_I, u'+v_I') - F_x(u,u')](v_r - v_{r-1}) + \\ &\qquad [F_y(u+v_I, u'+v_I') - F_y(u,u')](v_r' - v_{r-1}'), \end{aligned}$$

where $v_I = \nu v_r + (1-\nu)v_{r-1}$, $v_I' = \nu v_r' + (1-\nu)v_{r-1}'$ for some $\nu$, $0 \leq \nu \leq 1$. Applying the Mean Value Theorem again to each of the square brackets yields

$$\begin{aligned} G_r - G_{r-1} &= F_{xx}(v_r - v_{r-1})v_I + F_{xy}(v_r - v_{r-1})v_I' \\ &\qquad + F_{yx}(v_r' - v_{r-1}')v_I + F_{yy}(v_r' - v_{r-1}')v_I', \end{aligned}$$

where $F_{xx}, F_{xy}$ and $F_{yx}, F_{yy}$ are evaluated at certain intermediate points. If $(t, u+v_r, u'+v_r', \varepsilon)$ and $(t, u+v_{r-1}, u'+v_{r-1}', \varepsilon)$ are points in $\mathfrak{D}_\delta$, these intermediate points will also be in $\mathfrak{D}_\delta$, and by assumptions (B) and (E) there is a constant $k$ such that $4|F_{xx}|, \ 4|F_{xy}|, \ 4\varepsilon^{-1}|F_{yy}| \leq k$. Since $|v_I| < \max(|v_r|, |v_{r-1}|)$ and $|v_I'| \leq \max(|v_r'|, |v_{r-1}'|)$, we deduce that

(4.7) $$|G_r - G_{r-1}| \leq k\,\sigma$$

where/

where $\sigma$ is the larger of the two quantities

$$|v_r-v_{r-1}| \max(|v_{r-1}|, |v_r|, |v_{r-1}'|, |v_r'|)$$

and

$$|v_r'-v_{r-1}'| \max(|v_{r-1}|, |v_r|, \varepsilon|v_{r-1}'|, \varepsilon|v_r'|) .$$

We put $D_3 = 2D_1D_2k$ and shall prove by an induction that for sufficiently small $\varepsilon$

(4.8) $|v_n|, |v_n'| \leq 2D_1\eta$ ; $|v_n-v_{n-1}|, |v_n'-v_{n-1}'| \leq D_1\eta(D_3\eta)^n$

for all $n$. Since $\eta \to 0$ as $\varepsilon \to 0+$ we can choose $\varepsilon_1 \leq \varepsilon_0$ such that $2D_1\eta < \frac{\delta}{2}$ and $D_3\eta < \frac{1}{2}$ for $0 < \varepsilon < \varepsilon_1$.

It is obvious from (4.5) that (4.8) holds for $n = 0$. Assume it holds up to and including $n$. Then $(t, u+v_r, u'+v_r', \varepsilon)$ is in $\mathfrak{D}_\delta$ for $0 \leq r \leq n$, and we can form $G_r$, $0 \leq r \leq n$. Then, by (4.7), $|G_n - G_{n-1}| \leq 2k(D_1\eta)^2(D_3\eta)^n$, and hence by (4.6)

$|v_{n+1}-v_n| = |T_1G_n-T_1G_{n-1}| \leq D_1\eta(D_3\eta)^{n+1}$ since $D_3 = 2D_1D_2k$.

Similarly $|v_{n+1}'-v_n'| \leq D_1\eta(D_3\eta)^{n+1}$. Further

$|v_{n+1}| = \sum_{r=0}^{n+1}(v_r-v_{r-1})| \leq \sum_0^{\infty} D_1\eta(D_3\eta)^r \leq 2D_1\eta$ (for all $n$) since $D_3\eta < \frac{1}{2}$, and also by a similar argument $|v_{n+1}'| \leq 2D_1\eta$. Thus if (4.8) holds up to and including $n$, it holds for $n+1$, i.e. it holds for all $n$.

It follows from (4.8) that $v(t,\varepsilon) = \sum_{n=0}^{\infty}(v_n(t,\varepsilon)-v_{n-1}(t,\varepsilon)) = \lim_{n\to\infty} v_n(t,\varepsilon)$ defines a continuously differentiable function $v$.

$\lim_{n\to\infty} G_n(t,\varepsilon) = G(t, v(t,\varepsilon), v'(t,\varepsilon), \varepsilon)$ uniformly for $0 \leq t \leq 1$ by the continuity of $G$ in $v$ and $v'$, and the uniform convergence of $v_n$ to $v$ and $v_n'$ to $v'$. Similarly using the estimates above we deduce $T_1G_n \to T_1G$, so that $v$ satisfies the functional equation $v = T_1G$, and hence the differential equation (4.2) and the boundary conditions (4.3). Clearly $|v|, |v'| \leq 2D_1\eta$.

With this $v$, $x^* = u+v$ is a solution of the boundary value problem (4.1), and for $0 \leq t \leq 1$, and $0 < \varepsilon < \varepsilon_1$, the point $(t, x^*(t,\varepsilon), x^{*\prime}(t,\varepsilon), \varepsilon)$ is in $\mathfrak{D}_{\delta/2}$.

<u>The/</u>

The Inner Correction

We now seek a solution of $P_\varepsilon$ in the form $P_\varepsilon$

$$x = x^* + w = u + v + w .$$

Here $w$ satisfies the non-linear differential equation

$$(4.9) \qquad \varepsilon w'' + p^* w' + q^* w = G^*(t,w,w',\varepsilon)$$

and the boundary conditions

$$(4.10) \quad \begin{cases} a_1 w(0,\varepsilon)+\varepsilon a_2 w'(0,\varepsilon) = \alpha(\varepsilon) - a_1 x^*(0,\varepsilon) - \varepsilon a_2 x^{*\prime}(0,\varepsilon) = \mu(\varepsilon) = O(\eta) , \\ b_1 \ w(1,\varepsilon) + b_2 \ w'(1,\varepsilon) = 0 , \end{cases}$$

where $p^* = p^*(t,\varepsilon) = F_y(t,x^*,x^{*\prime},\varepsilon)$ ,

$q^* = q^*(t,\varepsilon) = F_x(t,x^*,x^{*\prime},\varepsilon)$ ,

and $G^*(t,w,w',\varepsilon) = G^*(w,w')$

$$= -H^*(t,\varepsilon) - \{F(x,x') - F(x^*,x^{*\prime}) - F_y(x^*,x^{*\prime})w' - F_x(x^*,x^{*\prime})w\} .$$

Now $F_y(u+v,\ u'+v') = F_y(u,u') + vF_{xy}(u+v_I,\ u'+v_I') + v' \ F_{yy}(u+v_I,\ u'+v_I')$

where $0 < |v_I| < |v|$ . Since $|v|, |v'| \leqslant 2D_1\eta$ and $\eta = O(\varepsilon)$ , we deduce that $p^* = \phi' + \varepsilon p_1^*$ with $\phi'$ the same function as in the previous section and assumption (C). Also

$$F_x(u+v,u'+v') = F_x(u,u') + v \ F_{xy}(u+v_J,u'+v_J') + v' \ F_{yy}(u+v_J,\ u'+v_J')$$

where $0 < v_J < v$ , so that $q^*(1,\varepsilon) = q^*(1,\varepsilon) + O(\eta)$ .

Thus $\{b_1 - b_2 \ q(1,\varepsilon)/\phi'(1,\varepsilon)\}^{-1}$ being $O(1)$ under assumption (F) implies $\{b_1 - b_2 \ q^*(1,\varepsilon)/\phi'(1,\varepsilon)\}^{-1} = O(1)$. Thus $p^*, q^*$ and $p_1^*$ satisfy the conditions imposed on $p, q$ and $p_1$ in chapter 2. With $p^*$, $q^*$ and the appropriate solutions of the linear differential equation.

$$V^{*\prime\prime} + p^* V^{*\prime} + q^* V^* = 0, \qquad 0 \leqslant t \leqslant 1,$$

corresponding to (2.1), we form $T^*_{2\mu}$ as in (2.16) and (2.18), where $\mu$ is given by (4.10) and is $O(\eta)$ .

Thus the boundary value problem (4.9), (4.10) is equivalent to the functional equation

(4.11)/

$$(4.11) \qquad w = T_2{}^* \, G^* .$$

We construct $w$ by successive approximation, through a sequence $w_n$ with $w_{-1} \equiv 0$ and $w_n = T_2{}^* \, G^*_{n-1}$ for $n = 0,1,2,\ldots$, where $G^*_n = G^*_n(t,\varepsilon) = G^*(t, w_n(t,\varepsilon) w_n{}'(t,\varepsilon), \varepsilon)$. Here

$$(4.12) \qquad G^*_{-1} = G^*(t,0,0,\varepsilon) = -H^*(t,\varepsilon) = O(\varepsilon^{-1}\eta \, e^{-m\phi/\varepsilon}) .$$

It can be proved by successive applications of the mean value theorem as in the construction of the outer correction $v$, that

$$(4.13) \qquad | G^*_r - G^*_{r-1} | < k\sigma^*$$

where $\sigma^*$ is the larger of

$$| w_r - w_{r-1} | \max(|w_{r-1}|, |w_r|, |w_{r-1}{}'|, |w_r{}'|)$$

and

$$| w_r{}' - w_{r-1}{}' | \max(|w_{r-1}|, |w_r|, \varepsilon |w_{r-1}{}'|, \varepsilon |w_r{}'|) .$$

We consider firstly the case $m > 1$. From (4.12) and (4.10) we have using the estimates (2.20) and (2.21)

$$(4.14) \qquad | w_o | \leqslant D^*_1 \eta (e^{-\phi/\varepsilon} + \frac{1}{\varepsilon} e^{-\phi(1)/\varepsilon}), \; \varepsilon | w_o{}' | \leqslant D^*_1 \eta e^{-\phi/\varepsilon} .$$

We also deduce that for any $\mu$

$$| f(t,\varepsilon) - g(t,\varepsilon) | \leqslant C(\varepsilon) \, e^{-\phi/\varepsilon} \, (e^{-\phi/\varepsilon} + \frac{1}{\varepsilon} e^{-\phi(1)/\varepsilon})$$

implies

$$(4.15) \qquad | T^*_{2\mu} f - T^*_{2\mu} g | \leqslant D^*_2 \varepsilon \; C(\varepsilon)(e^{-\phi/\varepsilon} + \frac{1}{\varepsilon} e^{-\phi(1)/\varepsilon})$$

and $\qquad |(T^*_{2\mu} f - T^*_{2\mu} g)'| \leqslant D^*_2 \; C(\varepsilon) \, e^{-\phi/\varepsilon}$.

We set $D^*_3 = 2D^*_1 D^*_2 \, k$ and choose $\varepsilon^*_1$ such that

$$2D^*_1 \eta \, (1 + (e\phi(1))^{-1}) < \frac{\delta}{2} \quad \text{and} \quad D^*_3 \eta < \frac{1}{2} \qquad \text{for} \quad 0 < \varepsilon < \varepsilon^*_1 < \varepsilon_o .$$

We prove by induction that

$$(4.16) \quad \left. \begin{aligned} &|w_n| \leqslant 2D^*_1 \eta \, (e^{-\phi/\varepsilon} + \frac{1}{\varepsilon} e^{-\phi(1)/\varepsilon}) \; , \; |w_n{}'| \leqslant 2D^*_1 \eta \frac{1}{\varepsilon} e^{-\phi/\varepsilon} , \\ &|w_n - w_{n-1}| \leqslant D^*_1 \eta \, (D^*_3 \eta)^n (e^{-\phi/\varepsilon} + \frac{1}{\varepsilon} e^{-\phi(1)/\varepsilon}) \; , \\ &\text{and } \varepsilon | w_n{}' - w_{n-1}{}' | \leqslant D^*_1 \eta \, (D^*_3 \eta)^n \, e^{-\phi/\varepsilon} \quad \text{for all } n . \end{aligned} \right\}$$

On the induction assumption,

$$\begin{aligned} | G^*_n - G^*_{n-1} | &\leqslant k\sigma^* \\ &\leqslant 2k \, \varepsilon^{-1} (D^*_1 \eta)^2 (D^*_3 \eta)^n \, e^{-\phi/\varepsilon} (e^{-\phi/\varepsilon} + \frac{1}{\varepsilon} e^{-\phi(1)/\varepsilon}) , \end{aligned}$$

so/

so that $|T^*_{2\mu} G^*_n - T^*_{2\mu} G^*_{n-1}| \leq 2k\, D^*_2 (D^*_1 \eta)^2 (D^*_3 \eta)^n (e^{-\phi/\varepsilon} + \frac{1}{\varepsilon} e^{-\phi(1)/\varepsilon})$

$$\varepsilon |(T^*_{2\mu} G^*_n - T^*_{2\mu} G^*_{n-1})'| \leq 2k D^*_2 (D^*_1 \eta)^2 (D^*_3 \eta)^n e^{-\phi/\varepsilon}$$

by (4.15). Putting $D^*_3 = 2D^*_1 D^*_2 k$ and using the same arguments as in the proof of (4.8), we obtain (4.16).

Thus for $m > 1$ we have a solution $w$ of the boundary value problem (4.9), (4.10) such that

$$|w| \leq 2 D^*_1 \eta \, (e^{-\phi/\varepsilon} + \frac{1}{\varepsilon} e^{-\phi(1)/\varepsilon}),$$
$$\varepsilon |w'| \leq 2 D^*_1 \eta \, e^{-\phi/\varepsilon},$$

and since for $0 < \varepsilon < \min(\varepsilon_1, \varepsilon^*_1)$ the graph of $x^* = u + v$ is in $\mathfrak{D}_{\delta/2}$, the graph of $x = u + v + w$ is in $\mathfrak{D}_\delta$.

Where $m = 1$ the proof of the result is rather more complicated. Since $G^*_{-1} = O(\varepsilon^{-1} \eta \, e^{-\phi/\varepsilon})$ and $\mu = O(\eta)$, we now have (using (2.22))

(4.17) $\quad |w_o|, \quad \varepsilon |w_o'| \leq D^*_4 \eta \, [(1 + \frac{\phi}{\varepsilon}) e^{-\phi/\varepsilon} + \frac{1}{\varepsilon^2} e^{-\phi(1)/\varepsilon}]$

instead of (4.14). Also from (2.22) we deduce in this case for

$$|f - g| \leq C(\varepsilon) e^{-\phi/\varepsilon},$$

(4.18) $\quad |T^*_{2\mu} f - T^*_{2\mu} g|, \ \varepsilon |(T^*_{2\mu} f - T^*_{2\mu} g)'| \leq D^*_5 \, \varepsilon C(\varepsilon) [(1 + \frac{\phi}{\varepsilon}) e^{-\phi/\varepsilon} + \frac{1}{\varepsilon^2} e^{-\phi(1)/\varepsilon}]$

In the proof we shall use the inequality

(4.19) $\quad [(1 + \frac{\phi}{\varepsilon}) e^{-\phi/\varepsilon} + \frac{1}{\varepsilon^2} e^{-\phi(1)/\varepsilon}]^2 < h \, e^{-\phi/\varepsilon}$

where $h = \frac{4}{e} + \frac{2}{\phi(1)^2} X + \frac{1}{\phi(1)^4} Y$

and $X = \max (1+x) x^2 e^{-x}$, $Y = \max x^4 e^{-x}$ for $x \geq 0$.

We set $D^*_6 = 2hk\, D^*_4 D^*_5$ and choose $\varepsilon^*_2$ such that

$$2D^*_4 \eta (1 + 4(e\phi(1))^{-2}) < \frac{\delta}{2} \quad \text{and} \quad D^*_6 \eta < \tfrac{1}{2}$$

for $0 < \varepsilon < \varepsilon^*_2$ and prove by induction that

(4.20) $\quad |w_n|, \varepsilon |w_n'| \leq 2 D^*_4 \eta \, [(1 + \frac{\phi}{\varepsilon}) e^{-\phi/\varepsilon} + \frac{1}{\varepsilon^2} e^{-\phi(1)/\varepsilon}]$

$$|w_n - w_{n-1}|, \varepsilon |w_n' - w_{n-1}'| \leq D^*_4 \eta (D^*_6 \eta)^n [(1 + \frac{\phi}{\varepsilon}) e^{-\phi/\varepsilon} + \frac{1}{\varepsilon^2} e^{-\phi(1)/\varepsilon}]$$

for all $n$. In the induction process we use (4.19) to show that (4.20) implies

$$\begin{aligned} |G^*_n - G^*_{n-1}| &\leq k\sigma^* \\ &\leq 2k\, \varepsilon^{-1} (D^*_4 \eta)^2 (D^*_6 \eta)^n [(1 + \tfrac{\phi}{\varepsilon}) e^{-\phi/\varepsilon} + \tfrac{1}{\varepsilon^2} e^{-\phi(1)/\varepsilon}]^2 \\ &\leq 2hk\, \varepsilon^{-1} (D^*_4 \eta)^2 (D^*_6 \eta)^n e^{-\phi/\varepsilon}, \end{aligned}$$

and/

and hence

$$\left| T_2^* G_n^* - T_2^* G_{n-1}^* \right| , \ \varepsilon \left| (T_2^* G_n^* - T_2^* G_{n-1}^*)' \right|$$
$$\leqslant 2hk\, D_5^*(D_4^*\eta)^2(D_6^*\eta)^n \quad [(1+\tfrac{\phi}{\varepsilon})e^{-\phi/\varepsilon} + \tfrac{1}{\varepsilon^2}e^{-\phi(1)/\varepsilon}]$$
$$\leqslant \quad D_4^*\eta\,(D_6^*\eta)^{n+1}\ [(1+\tfrac{\phi}{\varepsilon})e^{-\phi/\varepsilon} + \tfrac{1}{\varepsilon}e^{-\phi(1)/\varepsilon}]$$

using (4.18), since $D_6^* = 2hk\, D_4^* D_5^*$ . Otherwise the proof of (4.20) closely resembles that of (4.8) .

Thus for $m = 1$ we have a solution $w$ of the boundary value problem (4.9), (4.10) such that

$$|w| , \ \varepsilon|w'| \ \leqslant \ 2D_4^*\eta\ [(1+\tfrac{\phi}{\varepsilon})e^{-\phi/\varepsilon} + \tfrac{1}{\varepsilon^2}e^{-\phi(1)/\varepsilon}] < \tfrac{\delta}{2} \ ,$$

and for $0 < \varepsilon < \min(\varepsilon_1, \varepsilon_2^*)$ the graph of $x = u + v + w$ is again in $\mathcal{D}_\delta$ .

We have now constructed a solution to $P_\varepsilon$ in the form $x = u + v + w$ for $m \geqslant 1$ . From the estimates for $v$ and $w$ we deduce that for $\varepsilon$ sufficiently small

$$|x - u| = O(\eta) \quad \text{and} \quad |x' - u'| = O(\eta\,\Phi)$$

where
$$\Phi = \Phi(t,\varepsilon) = 1 + \tfrac{1}{\varepsilon}e^{-\phi/\varepsilon} \qquad \text{if} \quad m > 1 \ ,$$
$$= 1 + \tfrac{1}{\varepsilon}(1 + \tfrac{\phi}{\varepsilon})\, e^{-\phi/\varepsilon} \quad \text{if} \quad m = 1 \ .$$

## Chapter 5

## An Illustrative Example

In this chapter we look at an example showing the construction of an approximate solution in the form of an asymptotic series, and the raison d'etre of the estimates (3.3) for H and H* and of the restrictions made in assumption (F). The example chosen is one that has been discussed by Willett [22] and by Erdélyi [9].

Consider the boundary value problem

$$(5.0) \qquad \varepsilon x'' + x' + x^{N+1} = 0, \qquad 0 \leq t \leq 1 ,$$

with

$$(5.1) \qquad a_1\, x(0,\varepsilon) + \varepsilon a_2\, x'(0,\varepsilon) = \alpha(\varepsilon)$$

$$(5.2) \qquad b_1\, x(1,\varepsilon) + b_2\, x'(1,\varepsilon) = \beta(\varepsilon)$$

where N is an integer independent of $\varepsilon$ and $\alpha$ and $\beta$ admit power series expansions in $\varepsilon$, $\alpha(\varepsilon) \sim \sum_{n=0}^{\infty} \alpha_n \varepsilon^n$, $\beta(\varepsilon) \sim \sum_{n=0}^{\infty} \beta_n \varepsilon^n$. Here $F(t,x,x',\varepsilon) = x' + x^{N+1}$, so that $F_{x'} = 1$ and we may choose $\phi(t) = t$ for all $u(t,\varepsilon)$ approximating to the solution of the problem.

We seek an approximate solution using the two-variable expansion method developed (independently) by Kevorkian and Cochran . In view of the fact that the method is relatively new we will give here a fuller description of the expansion procedure than might have otherwise been necessary. We pose $x(t,\varepsilon) \sim \sum_{n=0}^{\infty} x_n(t,\tau)\varepsilon^n$ where $\tau = \tau(t,\varepsilon)$ is a suitable function of t and $\varepsilon$. While the choice of $\tau$ for a given problem is not uniquely determined, it is essentially a boundary layer coordinate. As the boundary layer term in x in the above problem is $O(e^{-t/\varepsilon})$ a natural choice of $\tau$ in this case is simply $\tau = t/\varepsilon$. (This choice is re-inforced by the exact solutions in the cases N = -1 and N = 0). The substitution of the expansions for x, $\alpha$ and $\beta$ into (5.0), (5.1) and (5.2) yields

(5.3)/

$$(5.3) \quad \varepsilon \sum_0^\infty (x_{n,tt} + \frac{2}{\varepsilon} x_{n,t\tau} + \frac{1}{\varepsilon^2} x_{n,\tau\tau})\varepsilon^n + \sum_0^\infty (x_{n,t} + \frac{1}{\varepsilon} x_{n,\tau})\varepsilon^n + [\sum_0^\infty x_n \varepsilon^n]^{N+1} = 0,$$

$$(5.4) \quad a_1 \sum_0^\infty x_n(0,0)\varepsilon^n + \varepsilon a_2 \sum_0^\infty (x_{n,t}(0,0) + \frac{1}{\varepsilon} x_{n,\tau}(0,0))\varepsilon^n = \sum_0^\infty \alpha_n \varepsilon^n ,$$

$$(5.5) \quad b_1 \sum_0^\infty x_n(1,\tfrac{1}{\varepsilon})\varepsilon^n + b_2 \sum_0^\infty (x_{n,t}(1,\tfrac{1}{\varepsilon}) + \frac{1}{\varepsilon} x_{n,\tau}(1,\tfrac{1}{\varepsilon}))\varepsilon^n \sim \sum_0^\infty \beta_n \varepsilon^n .$$

Equating the coefficients of powers of $\varepsilon$ to zero, we obtain from (5.3) the partial differential equations

$$(5.6) \qquad x_{0,\tau\tau} + x_{0,\tau} = 0 ,$$

$$(5.7) \qquad x_{1,\tau\tau} + 2x_{0,t\tau} + x_{0,t} + x_{1,\tau} + x_0^{N+1} = 0 ,$$

and for $n \geq 1$,

$$(5.8) \qquad x_{n+1,\tau\tau} + x_{n+1,\tau} + x_{n-1,tt} + 2x_{n,t\tau} + x_{n,t} + \sum_{r_i}^{*} \frac{(N+1)!}{r_0!\, r_1! \ldots . r_n!} x_0^{r_0} x_1^{r_1} \ldots . x_n^{r_n} = 0 ,$$

where the summation over the $r_i$ is such that $\sum_{i=0}^{n} r_i = N+1$ and $\sum_{i=0}^{n} ir_i = n$. Proceeding similarly with (5.4) and (5.5), we deduce that the $x_n$ are subject to the boundary conditions

$$(5.9) \quad a_1 x_0(0,0) + a_2 x_{0,\tau}(0,0) = \alpha_0 ,$$

$$(5.10) \quad a_1 x_n(0,0) + a_2 x_{n,\tau}(0,0) + a_2 x_{n-1,t}(0,0) = \alpha_n , \quad n \geq 1,$$

$$(5.11) \quad b_2 x_{0,\tau}(1,\tfrac{1}{\varepsilon}) \sim 0 ,$$

$$(5.12) \quad b_1 x_n(1,\tfrac{1}{\varepsilon}) + b_2 x_{n,t}(1,\tfrac{1}{\varepsilon}) + b_2 x_{n+1,\tau}(1,\tfrac{1}{\varepsilon}) \sim \beta_n , \quad n \geq 0 .$$

Here we have anticipated the fact that the boundary condition at $t=0$ will be fitted exactly at each step in the approximation procedure, while at $t=1$ it will be fitted asymptotically $(\varepsilon \to 0+)$ i.e. we might re-write (5.11) and (5.12) as

$$(5.13) \qquad b_2 x_{0,\tau}(1,\infty) = 0,$$

$$(5.14) \qquad b_1 x_n(1,\infty) + b_2 x_{n,t}(1,\infty) + b_2 x_{n+1,\tau}(1,\infty) = \beta_n , \quad n \geq 0 .$$

Integration of (5.6) with respect to $\tau$ gives

$$x_0(t,\tau) = u_0(t) + v_0(t)\, e^{-\tau}$$

where/

where $u_o(t)$ and $v_o(t)$ are arbitrary functions of $t$, which we will determine later. We substitute $x_o(t,\tau)$ into (5.7) to obtain the differential equation for $x_1$

$$(5.15)\quad x_{1,\tau\tau} + x_{1,\tau} = -\left(u_o'(t) + u_o(t)^{N+1}\right) + \left(v_o'(t) - (N+1)u_o^N\, v_o\right)e^{-\tau}$$
$$- \sum_{r=2}^{N+1} \binom{N+1}{r} v_o^r\, u_o^{\,N+1-r}\, e^{-r\tau} .$$

On integration we obtain

$$x_1(t,\tau) = u_1(t) + v_1(t)e^{-\tau} - \left(u_o'(t) + u_o(t)^{N+1}\right)\tau - \left(v_o'(t) - (N+1)u_o^N v_o\right)\tau e^{-\tau}$$
$$+ \sum_{r=2}^{N+1} \frac{1}{r(1-r)} \binom{N+1}{r} v_o^r\, u_o^{\,N+1-r}\, e^{-r\tau}$$

where $u_1(t)$, $v_1(t)$ are arbitrary functions of $t$ to be chosen later. The boundedness of $x_1$ as $\tau \to \infty$ (i.e. $\varepsilon \to 0$) implies that the coefficient of $\tau$ in the above expression must be zero, i.e. $u_o'(t) + u_o(t)^{N+1} = 0$, while (5.14) with $n=0$ (after substitution for $x_o$ and $x_1$) requires that $b_1\, u_o(1) + b_2\, u_o'\,(1) = \beta_o$ .

Integrating the differential equation for $u_o$ we obtain

$$u_o(t) = (B^{-N} - N + Nt)^{-1/N}$$

where $B = u_o(1)$ is a root of $b_1 B - b_2 B^{N+1} = \beta_o$ . (If $N$ is even we need the further requirement that $B < (\frac{1}{N})^{1/N}$ .) The condition (5.13) is satisfied automatically, while (5.9) becomes

$$a_1(u_o(0) + v_o(0)) - a_2 v_o(0) = \alpha_o$$

which fixes the initial value of $v_o(t)$ providing $a_1 \neq a_2$ . Apart from its initial value the choice of $v_o(t)$ remains open. To 'simplify' subsequent steps in the expansion procedure, we eliminate from $x_1(t,\tau)$ the term in $\tau e^{-\tau}$, i.e. we choose $v_o(t)$ such that

$$v_o'(t) - (N+1)\, u_o(t)^N\, v_o(t) = 0$$

Integrating this equation remembering that the value of $v_o(t)$ at $t=0$ is already fixed, we obtain

$$v_o(t) = \left(\frac{\alpha_o - a_1 B^*}{a_1 - a_2}\right) \left(\frac{B^*}{u_o(t)}\right)^{N+1}$$

where $B^* = u_o(0) = (B^{-N} - N)^{-1/N}$ .

Thus/

Thus $x_o(t,\tau) = u_o(t) + v_o(t)\, e^{-\tau}$ is completely determined. We note in passing that the differential equations satisfied by $u_o$ and $v_o$ could have been found, without integrating the differential equation for $x_1$, by equating to zero the term independent of $\tau$ and the coefficient of $e^{-\tau}$ in the righthand side of (5.15).

The substitution for $x_o$ and $x_1$ in the equation for $x_2$ (5.8 with $n = 1$) gives

$$x_{2\tau\tau} + x_{2\tau} = \sum_{r=0}^{2N+1} C_{2,r}(t)\, e^{-r\tau}$$

where the $C_{2,r}$'s are independent of $\tau$ and in particular

$$C_{2,0} = - u_o'' - u_1' - (N+1)\, u_o^N u_1 \quad ,$$

$$C_{2,1} = -v_o'' + v_1' - (N+1)N\, u_1 v_o u_o^{N-1} - (N+1) u_o^N\, v_1 \ .$$

The elimination of the terms $\tau$ and $\tau e^{-\tau}$ (sometimes called the 'secular' terms) from $x_2$ implies that $C_{2,0}$ and $C_{2,1}$ are both zero. Thus $u_1$ satisfies the linear differential equation

$$u_o'' + u_1' + (N+1)\, u_o^N u_1 = 0 \quad , \quad \text{and}$$

$v_1$ satisfies the linear differential equation

$$v_1' - (N+1)\, u_o^N\, v_1 = N(N+1)\, u_1 v_o u_o^{N-1} + v_o'' \ .$$

Further $u_1$ and $v_1$ are completely determined by the boundary conditions imposed on $x_1$. (5.14) with $n = 1$ yields

$$b_1\, u_1(1) + b_2\, u_1'(1) = \beta_1 \quad ,$$

and substituting for $u_1'(1)$ we obtain

$$u_1(1) = C = \frac{\beta_1 - b_2(N+1)B^{2N+1}}{b_1 - b_2(N+1)B^N}$$

providing $b_1 - b_2(N+1)B^n \neq 0$. Integrating the differential equation for $u_1(t)$ we obtain

$$u_1(t) = u_o^{N+1} \left[ \frac{C}{B^{N+1}} + (N+1) \log \frac{u_o}{B} \right] .$$

(5.10) with $n = 1$ yields

$$a_1 u_1(0) + a_1 v_1(0) + a_1 \sum_2^{N+1} \frac{1}{r(1-r)} \binom{N+1}{r} v_o(0)^r\, u_o(0)^{N+1-r}$$

$$+ a_2 u_o'(0) + a_2 v_o'(0) - a_2 v_1(0) - a_2 \sum_2^{N+1} \frac{1}{1-r} \binom{N+1}{r} v_o(0)^r\, u_o(0)^{N+1-r} = \alpha_1 \ .$$

Solving/

Solving we obtain $v_1(0)$, and integrating the equation for $v_1(t)$ we obtain

$$v_1(t) = \frac{K}{u_o(t)^{N+1}} - \frac{(N+1)}{u_o(t)} \left(\frac{\alpha_o - a_1 B^*}{a_1 - a_2}\right) B^{*N+1}\left[\frac{C}{B^{N+1}} - 1 + (N+1)\log\frac{u_o}{B}\right]$$

where $K$ is a constant (such that $v_1 = v_1(0)$ at $t = 0$) .

Thus $x_1(t,\tau) = u_1(t) + v_1(t)e^{-\tau} + \sum_2^{N+1} \frac{1}{r(1-r)} \binom{N+1}{r} v_o^{\,r}\, u_o^{\,N+1-r}\, e^{-r\tau}$
where both $u_1(t)$ and $v_1(t)$ are completely determined.

We can proceed in this way to obtain $x_2(t,\tau), x_3(t,\tau), \ldots.,$ the complexity of the terms increases at each step. In general

$$x_n(t,\tau) = u_n(t) + v_n(t)\, e^{-\tau} + \sum_{r=2}^{nN+1} c_{nr}(t)\, e^{-r\tau}$$

where $u_n(t)$, $v_n(t)$ satisfy $(n \geq 1)$ linear differential equations

$$u''_{n-1} + u'_n + \sum_{r_i}{}^{*} \frac{(N+1)!}{r_o! r_1! \ldots . r_n!}\, u_o^{r_o} u_1^{r_1} \ldots . u_n^{r_n} = 0$$

with $b_1\, u_n(1) + b_2\, u'_n(1) = \beta_n$ , and

$$v''_{n-1} - v'_n + (N+1)\, u_o^N\, v_n + f(u_o, u_1, \ldots, u_n;\ v_o, v_1, \ldots v_{n-1}) = 0$$

with $v_n(0)$ chosen to satisfy (5.10) . ($f$ is the coefficient of $e^{-\tau}$ in $\left\{ \sum^{*} \frac{(N+1)!}{r_o! r_1! \ldots r_n!}\, x_o^{r_o} \ldots x_n^{r_n} - (N+1)\, u_o^N\, v_n\, e^{-\tau} \right\}$ and is independent of $v_n$ . )

We will show by a direct application of the theorem set out in chapter 3 that the series obtained in this way is such that

$$x(t,\varepsilon) = \sum_{n=o}^{m} x_n(t,\tau)\varepsilon^n + O(\varepsilon^{m+1}) ,$$

and $$x'(t,\varepsilon) = \sum_{n=o}^{m} x'_n(t,\tau)\ ^n + O(\varepsilon^{m+1}) + O(\varepsilon^m\, e^{-t/\varepsilon}) ,$$

for $0 \leqslant t \leqslant 1$, $0 \leqslant \varepsilon \leqslant \varepsilon_o$ . The assumptions are all met in this case, and in particular assumption (F) becomes

$$a_1 \neq a_2$$

and $b_1 - b_2\, (N+1)\, B^N \neq 0$ , both of which are met in the expansion procedure.

Let $X_m(t,\varepsilon)$ be the sum of the first $(m+1)$ terms of the series/

series obtained i.e. $X_m(t,\varepsilon) = \sum_{n=o}^{m} x_n(t,\tau)\varepsilon^n$ . Substitution into the differential equation for $x(t,\varepsilon)$ yields

$$\varepsilon X_m'' + X_m' + X_m^{n+1} = \frac{1}{\varepsilon}(x_{o,\tau\tau} + x_{o,\tau}) + (x_{1,\tau\tau} + x_{1,\tau} + 2x_{o,t\tau} + x_{o,t} + x_o^{N+1})$$

$$+ \sum_{n=1}^{m-1}(x_{n-1,tt} + 2x_{n,t\tau} + x_{n+1,\tau\tau} + x_{n+1,\tau} + x_{n,t} + \sum_{r_i}^{*} \frac{(N+1)!}{r_o!\ldots r_n!} x_o^{r_o}\ldots x_n^{r_n})\varepsilon^n$$

$$+(x_{m-1,tt} + 2x_{m,t\tau} + x_{m,t} + \sum_{r_i}^{*} \frac{(N+1)!}{r_o!\ldots r_m!} x_o^{r_o}\ldots x_m^{r_m})\varepsilon^m + O(\varepsilon^{m+1}) .$$

But $u_m(t)$, $v_m(t)$ are chosen so that the coefficient of $\varepsilon^m$ in the above is $O(e^{-2\tau})$, and using this in conjunction with (5.6),(5.7) and (5.8) with $n=1,2,\ldots,m-1$, we obtain

$$\varepsilon X_m'' + X_m' + X_m^{N+1} = O(\varepsilon^m e^{-2t/\varepsilon}) + O(\varepsilon^{m+1}) .$$

Because of our choice of $u_n(1)$ and $v_n(0)$ in the expansion procedure we also have

$$\alpha(\varepsilon) - a_1 X_m(0,\varepsilon) - \varepsilon a_2 X_m' (0,\varepsilon) = O(\varepsilon^{m+1}) ,$$

$$\beta(\varepsilon) - b_1 X(1,\varepsilon) - b_2 X_m' (1,\varepsilon) = O(\varepsilon^{m+1}) .$$

Thus applying the theorem with $\eta = \varepsilon^{m+1}$

$$x(t,\varepsilon) = X_m(t,\varepsilon) + O(\varepsilon^{m+1}) ,$$

and $x'(t\ \varepsilon) = X_m' (t,\varepsilon) + O(\varepsilon^{m+1}) + O(\varepsilon^m e^{-t/\varepsilon})$ .

By a similar analysis we find the same estimates hold for the series obtained when the two-variable method outlined above is applied to differential equations of the type

$$\varepsilon x'' + (1 + \varepsilon h(x,t,\varepsilon))x' + g(x,t,\varepsilon) = 0$$

where $x$ satisfies boundary conditions of the type (5.1),(5.2), and $g$ and $h$ admit asymptotic expansions in powers of $\varepsilon$ which are uniform in $x$ and $t$ and whose terms are regular in $x$ and $t$ . The computations in the general case are of course much heavier.

Returning to the specific problem (5.0),(5.1),(5.2), we see that when $a_2 = b_2 = 0$ , $a_1 = b_1 = 1$ , and $\alpha$ and $\beta$ are independent of $\varepsilon$ (so that $\alpha_o = \alpha$, $\beta_o = \beta$), the first approximation obtained

by/

by the two-variable procedure above is exactly that obtained by Willett [22]. As Erdélyi [9] shows it one of the many possible first approximations. Neither Erdélyi nor Willett consider higher approximations.

## Acknowledgement

I wish to thank Professor Erdélyi for his helpful and friendly advice in the course of this work, and for the patience and care he exercised in reading the manuscript.

## References


1 Cochran, J.A., Problems in singular perturbation theory, Technical report no. 1, Applied Mathematics and Statistics Laboratories, Stanford University, California, 1962.

2 Coddington, E.A. and Levinson, N., A boundary value problem for a nonlinear differential equation with a small parameter, Proc. Amer. Math. Soc., 3, 73-81, 1952.

3 Cole, J.D. and Kevorkian, J., Uniformly valid asymptotic approximations for certain non-linear differential equations, Proc. Internat. Sympos. Non-linear Differential Equations and Non-linear Mechanics, Academic Press, New York 1963,pp.113-120.

4 Cole, J.D., Perturbation methods in applied mathematics, Ginn-Blaisdell, Waltham, 1968.

5 Eckhaus, W., On the foundation of the method of matched asymptotic expansions, J. Méc., 8, 265-300, 1969.

6 Erdélyi, A., An expansion procedure for singular perturbations, Atti Acad. Sci. Torino, Cl. Sci. Fis. Mat. Natur. 95, 651-672, 1961.

7 Erdélyi, A., Singular perturbations, Bull. Amer. Math. Soc., 68, 420-424, 1962.

8 Erdélyi, A., On a non-linear boundary value problem involving a small parameter, J. Australian Math. Soc., 2, 425-439, 1962.

9 Erdélyi, A., Approximate solutions of a nonlinear boundary value problem, Arch. Rational Mech. Anal., 29, 1-17, 1968.

10 Fraenkel, L.E., On the method of matched asymptotic expansions, Part I: A matching principle, Part II: Some applications of the composite series, Part III: Two boundary-value problems, Proc. Camb. Phil. Soc. 65 , 209-231, 233-261, 263-284, 1969.

11 Haber, S. and Levinson, N., A boundary value problem for a singularly perturbed differential equation, Proc. Amer. Math. Soc., 6, 866-872, 1955.

12 Harris, W.A. Jr., Singular perturbations on a boundary value problem for a non-linear system of differential equations, Duke Math. J. 29, 429-446, 1962.

13 Kaplun, S. and Lagerstrom, P.A., Asymptotic expansions of Navier-Stokes solutions for small Reynolds numbers, J. Math. Mech. 6, 585-593, 1957.

14 Kevorkian, J., The two-variable expansion procedure for the approximate solution of certain non-linear differential equations, Lectures in Applied Mathematics, Vol.7, Space Mathematics, Pt.3, 206-275, 1966.

15 Krylov, N.M. and Bogoliubov, N.N., Introduction to Nonlinear Mechanics, Acad. Sci. Ukrain. S.S.R. (1937), translated by S. Lefschetz, Princeton Univ. Press, Princeton, 1947.


16/

16 Lagerstrom, P.A., Some recent developments in the theory of singular perturbations, "Symposium on Problems in Analysis in Honor of Salomon Bochner" Princeton University, 1969.

17 Macki, J.W., Singular perturbations of a boundary value problem for a system of nonlinear ordinary differential equations, Arch. Rational Mech. Anal. 24, 219-232, 1967.

18 O'Malley, R.E., Jr., A boundary value problem for certain non-linear second order differential equations with a small parameter, Arch. Rational Mech. Anal. 29, 66-74, 1968.

19 Van Dyke, M., Perturbation methods in fluid dynamics, Academic Press, New York, 1964.

20 Wasow, W., Singular perturbation of boundary value problems for nonlinear differential equations of the second order, Comm. Pure Appl. Math. 9, 93-113, 1956.

21 Willett, D., Nonlinear vector integral equations as contraction mappings, Arch. Rational Mech. Anal. 15, 79-86, 1964.

22 Willett, D., On a non-linear boundary value problem with a small parameter multiplying the highest derivative, Arch. Rational Mech. Anal. 23, 276-287, 1966.